Example 2.21a $\lim_{f\to\infty}\left(\frac{3+2f}{2f}\right)^{f+3}$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$= \lim_{f\to\infty}\left(\frac{3}{2f}+\frac{2f}{2f}\right)^{f+3}$$

$$= \lim_{f\to\infty}\left(1+\frac{3}{2f}\right)^{f+3}$$

$$= \lim_{f\to\infty}\left(1+\frac{1}{2f/3}\right)^{f+3} = \lim_{f\to\infty}\left(1+\frac{1}{2f/3}\right)^{\left(\frac{2f}{3}\right)\boxed{\left(\frac{3}{2f}\right)(f+3)}}$$

$$= \lim_{f\to\infty}\left[\underbrace{\left(1+\frac{1}{2f/3}\right)^{\frac{2f}{3}}}_{e}\right]^{\boxed{\frac{3(f+3)}{2f}}}$$

$$\lim_{f\to\infty} \frac{3(f+3)}{2f} = \frac{3\cancel{f}\left(1+\frac{3}{f}\right)}{2\cancel{f}} = \frac{3}{2}$$

$$= \boxed{e^{3/2}}$$

Alternately

$$\lim_{f\to\infty}\frac{3(f+3)}{2f} = \lim_{f\to\infty}\frac{3f+9}{2f}$$

$$= \lim_{f\to\infty}\left(\frac{3\cancel{f}}{2\cancel{f}}+\underbrace{\frac{9}{2f}}_{0}\right) = \boxed{\frac{3}{2}}.$$

2.7

2.7 Consider the functions below. In each case, determine if the function is continuous at $x = 1$. Justify your answers.

(a) $A(x) = \begin{cases} x & x < 1 \\ 3 & x = 1 \\ 2x-1 & x > 1 \end{cases}$

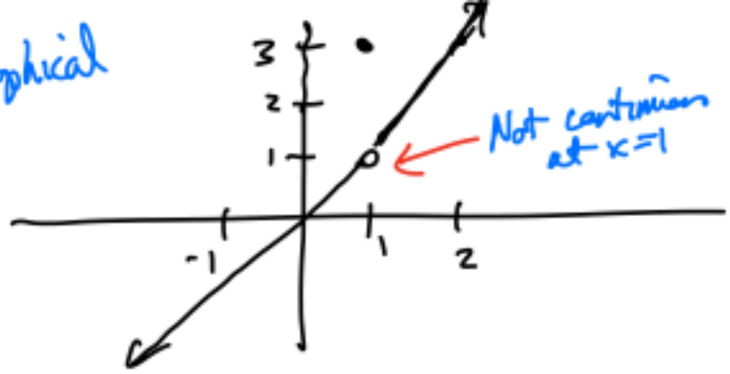
Algebra

$\lim_{x\to 1^+} A(x) = \lim_{x\to 1^+} 2x-1 = 1$

$\lim_{x\to 1^-} A(x) = \lim_{x\to 1^-} x = 1$

$A(1) = 3$ Not continuous at $x=1$.

Graphical



Numerical

x	$A(x)$
.9	.9
.99	.99
1	3
1.001	1.002
1.1	1.2

a jump! Not continuous at $x=1$.

Example: By using a limit of slopes, find the derivative of $g(x) = \frac{1}{\sqrt{x}}$ at $x=4$.

[Do you expect the answer to be positive or negative?]

positive 2 | negative 19 | zero 0

makes sense because as x increases, $\frac{1}{\sqrt{x}}$ gets smaller, so slope should be downward.

$$g'(4) = \frac{dg}{dx}(4) = \lim_{h\to 0} \frac{g(4+h) - g(4)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2}{2\sqrt{4+h}} - \frac{1\sqrt{4+h}}{2\sqrt{4+h}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}}{h/1} = \lim_{h\to 0} \frac{(2-\sqrt{4+h})}{2\sqrt{4+h}} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{(2-\sqrt{4+h})(2+\sqrt{4+h})}{(2h\sqrt{4+h})(2+\sqrt{4+h})}$$

$$(A-B)(A+B) = A^2 - B^2$$

$$= \lim_{h\to 0} \frac{4-(4+h)}{2h\sqrt{4+h}\,(2+\sqrt{4+h})} = \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-\cancel{h}^{\,1}}{2\cancel{h}\sqrt{4+h}\,(2+\sqrt{4+h})}$$

$$= \lim_{h\to 0} \frac{-1}{2\sqrt{4+h}\,(2+\sqrt{4+h})} = \frac{-1}{2\sqrt{4}\,(2+\sqrt{4})} = \frac{-1}{2\cdot 2(2+2)}$$

$$= \boxed{\frac{-1}{16}}$$

Formulas for the derivative.

① **Power rule**

$$\boxed{(x^n)' = n x^{n-1}}$$

(works for any real number n).

i.e. if $\phi(x) = x^7$, then

$$\phi'(x) = 7x^6$$

If $B(x) = \sqrt{x} = x^{1/2}$, then

$$B'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

$$\therefore B'(4) = \frac{1}{2\cdot\sqrt{4}} = \frac{1}{2\cdot 2} = \frac{1}{4}.$$

Let $F(x) = x^n$ — n fixed.

I will assume it is a positive integer & prove in that case.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Question: What is $(x+h)^n$?

Pascal's triangle: